Elements of Proof — Two-regimes interference classifier: an interference aware resource allocation algorithm

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***Abstract*—The following paper contains informative elements of proof related to the conference paper "'Title to be defined for 2-Regimes Interference Classifier"', submitted to WCNC 2014.**

I. Appendix: $(1,2)$ and $(2,1)$ are outperformed

Our objective is to show, that, for any SNR/INR configuration, the regimes (1,2) and (2,1) are outperformed by either (1,1), (2,2), (1,3) or (3,1), when focusing on the total spectral efficiency. In the following, we denote by L, the following term:

$$\epsilon_{(1,1)} = L = \log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right) + \log_2\left(1+\frac{\gamma_2}{1+\delta_2}\right)$$

Proposition I.1. *$(2,2) \rhd (1,2)$ is equivalent to $L < \log_2(1+\gamma_2)$*

Proof. The maximal achievable spectral efficiency for the regime (1,2) is

$$\epsilon_{(1,2)} = \frac{1}{2}L + \frac{1}{2}\log_2(1+\gamma_1)$$

The maximal achievable spectral efficiency for the regime (2,2) is

$$\epsilon_{(2,2)} = \frac{1}{2}\log_2(1+\gamma_1) + \frac{1}{2}\log_2(1+\gamma_2)$$

Then, $(2,2) \rhd (1,2)$ is equivalent to $\epsilon_{(1,2)} \leq \epsilon_{(2,2)}$, which immediately leads to $L < \log_2(1+\gamma_2)$. □

Proposition I.2. *$(2,2) \rhd (2,1)$ is equivalent to $L \leq \log_2(1+\gamma_1)$*

Proof. By symmetry, from Proposition I.1. □

Proposition I.3. *$(1,1) \rhd (1,2)$ is equivalent to $L > \log_2(1+\gamma_1)$*

Proof. By definition, $(1,1) \rhd (1,2)$ is equivalent to $\epsilon_{(1,2)} \leq \epsilon_{(1,1)}$, which immediately leads to:

$$L \geq \log_2(1+\gamma_2)$$

□

Proposition I.4. *$(1,1) \rhd (2,1)$ is equivalent to $L \geq \log_2(1+\gamma_1)$*

Proof. By symmetry, from Proposition I.3. □

From the previous four propositions, we enlisted in table I, the best regime among (1,1), (1,2), (2,1) and (2,2) for every possible configuration.

TABLE I
SUMMARY OF BEST REGIME FOR EACH CONFIGURATION

	$L > \log_2(1+\gamma_1)$	$L < \log_2(1+\gamma_1)$
$L > \log_2(1+\gamma_2)$	$(1,1)$	$(1,2)$
$L < \log_2(1+\gamma_2)$	$(2,1)$	$(2,2)$

From the previous table, we can state that (1,2) and (2,1) are outperformed by either (1,1) or (2,2), except when $\log_2(1+\gamma_2) > L > \log_2(1+\gamma_1)$ or $\log_2(1+\gamma_2) < L < \log_2(1+\gamma_1)$. In the following, we focus on the scenario, consisting of $\log_2(1+\gamma_2) > L > \log_2(1+\gamma_1)$, which leads to the 'a priori' best configuration (1,2).

Proposition I.5. *When $\log_2(1+\gamma_1) > L > \log_2(1+\gamma_2)$, $(3,1) \rhd (1,2)$.*

Proof. The maximal achievable spectral efficiency for the regime (3,1) is

$$\epsilon_{(3,1)} = \log_2(1+\gamma_1) + \log_2\left(1+\min\left[\frac{\gamma_2}{1+\delta_2}, \frac{\delta_1}{1+\gamma_1}\right]\right)$$

We compute

$$\epsilon_{(3,1)} - \epsilon_{(1,2)} = \frac{1}{2}\left[\log_2(1+\gamma_1) - L\right] + \log_2\left(1+\min\left[\frac{\gamma_2}{1+\delta_2}, \frac{\delta_1}{1+\gamma_1}\right]\right)$$

Since $\log_1(1+\gamma_2) > L$, necessarily $\epsilon_{(3,1)} - \epsilon_{(1,2)} > 0$, which means that $(3,1) \rhd (1,2)$. □

Proposition I.6. *When $\log_2(1+\gamma_1) < L < \log_2(1+\gamma_2)$, $(1,3) \rhd (2,1)$.*

Proof. From I.5, symmetric configuration. □

According to the previous propositions, we can finally state that, for any configuration, (1,2) and (2,1) are outperformed either by (1,1), (2,2), (3,1) or (1,3).

II. Appendix: $(2,3)$ and $(3,2)$ are outperformed

Our objective is to show, that, for any SNR/INR configuration, the regimes (3,2) and (2,3) are outperformed, in terms of total spectral efficiency, respectively by (3,1) and (1,3). Since the regimes (2,3) and (3,2) are symmetric, we focus only on the regime (2,3).

The maximal achievable spectral efficiency for the regime (2,3) is

$$\epsilon_{(2,3)} = \log_2(1+\gamma_2)+\min\left[\frac{1}{2}\log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right),\log_2\left(\frac{\delta_2}{1+\gamma_2}\right)\right]$$

The maximal achievable spectral efficiency for the regime (3,1) is

$$\epsilon_{(1,3)} = \log_2(1+\gamma_2)+\min\left[\log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right),\log_2\left(\frac{\delta_2}{1+\gamma_2}\right)\right]$$

Since $\min\left[\log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right),\log_2\left(\frac{\delta_2}{1+\gamma_2}\right)\right] \geq \min\left[\frac{1}{2}\log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right),\log_2\left(\frac{\delta_2}{1+\gamma_2}\right)\right]$, it comes immediately that $\epsilon_{(2,3)} \leq \epsilon_{(1,3)}$, which means $(1,3) \triangleright (2,3)$.

III. Appendix: $(2,2)$ is outperformed by either $(1,3)$ or $(3,1)$

In this section, our objective is to show that (2,2) is always outperformed by either (1,3) or (3,1). For example, the maximal achievable spectral efficiency for the regime (2,2) is

$$\epsilon_{(2,2)} = \frac{1}{2}\log_2(1+\gamma_1)+\frac{1}{2}\log_2(1+\gamma_2)$$

The maximal achievable spectral efficiency for the regime (3,1) is

$$\epsilon_{(3,1)} = \log_2(1+\gamma_1)+\min\left[\log_2(1+\gamma_2),\log_2\left(\frac{\delta_1}{1+\gamma_1}\right)\right]$$

We can state that, for any configuration

$$\epsilon_{(3,1)}-\epsilon_{(2,2)} = \frac{1}{2}\left[\log_2(1+\gamma_1)-\log_2(1+\gamma_2)\right] + \min\left[\log_2(1+\gamma_2),\log_2\left(\frac{\delta_1}{1+\gamma_1}\right)\right]$$

A sufficient condition to $\epsilon_{(3,1)}-\epsilon_{(2,2)} \geq 0$, i.e $(3,1)\triangleright(2,2)$ is $\gamma_1 > \gamma_2$.

Similarily, we can show that a sufficient condition to $(1,3)\triangleright(2,2)$ is $\gamma_1 < \gamma_2$. From both previous statements, we can conclude that, for any configuration, $(3,1)\triangleright(2,2)$ or $(1,3)\triangleright(2,2)$.

IV. Appendix: Conditions for $(1,1)\triangleright(1,3)$ and $(1,1)\triangleright(3,1)$

Let us first recall the performance of the three considered regimes, given by:

$$\epsilon_{(1,1)} = \log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right)+\log_2\left(1+\frac{\gamma_2}{1+\delta_2}\right)$$

$$\epsilon_{(1,3)} = \log_2\left(1+\min\left[\frac{\gamma_1}{1+\delta_1},\frac{\delta_2}{1+\gamma_2}\right]\right)+\log_2(1+\gamma_2)$$

$$\epsilon_{(3,1)} = \log_2\left(1+\min\left[\frac{\gamma_2}{1+\delta_2},\frac{\delta_1}{1+\gamma_1}\right]\right)+\log_2(1+\gamma_1)$$

In this appendix, we demonstrate the following proposition

Proposition IV.1.

$$(1,1)\triangleright(1,3) \Leftrightarrow \gamma_1 \geq \delta_2(1+\delta_1)$$

$$(1,1)\triangleright(3,1) \Leftrightarrow \gamma_2 \geq \delta_1(1+\delta_2)$$

Proof. Let us first consider C defined by

$$C_1 = \log_2\left(1+\min\left[\frac{\gamma_1}{1+\delta_1},\frac{\delta_2}{1+\gamma_2}\right]\right)+\log_2(1+\gamma_2) - \log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right)-\log_2\left(1+\frac{\gamma_2}{1+\delta_2}\right)$$

Firstly, by definition of the $\triangleright$ operator, we state that $(1,3)\triangleright(1,1) \Leftrightarrow C \geq 0$. One can easily verify that $\frac{\delta_2}{1+\gamma_2} \geq \frac{\gamma_1}{1+\delta_1}$ is a sufficient condition for $(1,3)\triangleright(1,1)$. From this, we deduce that $\frac{\gamma_1}{1+\delta 1} \geq \frac{\delta_2}{1+\gamma_2}$ is a necessary, yet not sufficient, condition for $(1,1)\triangleright(1,3)$.

Assuming $\frac{\delta_2}{1+\gamma_2} \leq \frac{\gamma_1}{1+\delta_1}$, C rewrites:

$$C' = \log_2\left(1+\frac{\delta_2}{1+\gamma_2}\right)+\log_2(1+\gamma_2) - \log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right)-\log_2\left(1+\frac{\gamma_2}{1+\delta_2}\right)$$

Which rewrites:

$$C' = \log_2(1+\delta_2)-\log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right)$$

In this configuration, one can verify that $C' \leq 0$ is equivalent to $\frac{\gamma_1}{1+\delta_1} > \delta_2$. In the end, we get that $(1,1)\triangleright(1,3) \Leftrightarrow \left[\frac{\gamma_1}{1+\delta_1} \geq \delta_2 \text{ and } \frac{\gamma_1}{1+\delta_1} \geq \frac{\delta_2}{1+\gamma_2}\right]$, which immediately leads to $(1,1)\triangleright(1,3) \Leftrightarrow \frac{\gamma_1}{1+\delta_1} \geq \delta_2$.

The same way, we can prove that $(1,1)\triangleright(3,1) \Leftrightarrow \gamma_2 \geq \delta_1(1+\delta_2)$ holds. □

V. Appendix: Conditions for $(1,3) \triangleright (3,3)$ and $(3,1) \triangleright (3,3)$

Let us first recall the performances for each regime

$$\epsilon_{(1,3)} = \log_2\left(1+\min\left[\frac{\gamma_1}{1+\delta_1}, \frac{\delta_2}{1+\gamma_2}\right]\right) + \log_2(1+\gamma_2)$$

$$\epsilon_{(3,1)} = \log_2\left(1+\min\left[\frac{\gamma_2}{1+\delta_2}, \frac{\delta_1}{1+\gamma_1}\right]\right) + \log_2(1+\gamma_1)$$

$$\begin{aligned}\epsilon_{(3,3)} &= \log_2\left(1+\min\left[\gamma_1, \tfrac{\delta_2}{1+\gamma_2}\right]\right) \\ &+ \log_2\left(1+\min\left[\gamma_2, \tfrac{\delta_1}{1+\gamma_1}\right]\right)\end{aligned}$$

We focus on defining a criterion based on $\gamma_1, \gamma_2, \delta_1, \delta_2$ for $(1,3) \triangleright (3,3)$. In a symmetrical way, one can deduce a criterion for $(3,1) \triangleright (3,3)$, based on $(1,3) \triangleright (3,3)$. Let us also define C_{13-33} as:

$$C_{13-33} = \log_2(1+\gamma_2) + \log_2\left(1+\min\left[\frac{\gamma_1}{1+\delta_1}, \frac{\delta_2}{1+\gamma_2}\right]\right) - \log_2\left(1+\min\left[\gamma_1, \frac{\delta_2}{1+\gamma_2}\right]\right) - \log_2\left(1+\min\left[\gamma_2, \frac{\delta_1}{1+\gamma_1}\right]\right)$$

And $(1,3) \triangleright (3,3) \Leftrightarrow C_{13-33} \geq 0$.

Proposition V.1. *A sufficient condition for $(1,3) \triangleright (3,3)$ is $\gamma_2 \geq \delta_1$. A sufficient condition for $(3,1) \triangleright (3,3)$ is $\gamma_1 \geq \delta_2$.*

Proof. If $\gamma_2 \leq \delta_1 \leq \frac{\delta_1}{1+\gamma_1}$, C_{13-33} rewrites

$$\begin{aligned}C_{13-33} &= \log_2(1+\gamma_2) + \log_2\left(1+\min\left[\frac{\gamma_1}{1+\delta_1}, \frac{\delta_2}{1+\gamma_2}\right]\right) \\ &- \log_2\left(1+\min\left[\gamma_1, \frac{\delta_2}{1+\gamma_2}\right]\right) - \log_2\left(1+\frac{\delta_1}{1+\gamma_1}\right)\end{aligned}$$

At this point, we may distinguish 3 cases.

1) $\gamma_1 \leq \frac{\delta_2}{1+\gamma_2}$
2) $\gamma_1 \geq \frac{\gamma_1}{1+\delta_1} \geq \frac{\delta_2}{1+\gamma_2}$
3) $\gamma_1 \geq \frac{\delta_2}{1+\gamma_2} \geq \frac{\gamma_1}{1+\delta_1}$

- When $\gamma_1 \leq \frac{\delta_2}{1+\gamma_2}$, we get $C_{13-33} \geq 0 \Leftrightarrow \gamma_2 \geq \delta 1$, which is true, by definition.
- When $\gamma_1 \geq \frac{\gamma_1}{1+\delta_1} \geq \frac{\delta_2}{1+\gamma_2}$, we get $C_{13-33} \geq 0 \Leftrightarrow \gamma_2 \geq \frac{\delta_1}{1+\gamma_1}$, which is also true, by definition.
- Finally, when $\gamma_1 \geq \frac{\delta_2}{1+\gamma_2} \geq \frac{\gamma_1}{1+\delta_1}$, a sufficient condition for $C_{13-33} \geq 0$ is $\frac{\delta_2}{1+\gamma_2} \geq \frac{\gamma_1}{1+\delta_1}$ and $\gamma_2 \geq \delta_1$, which is also true, by definition.

When, then conclude, that $\gamma_2 \leq \delta_1$ is a sufficient condition for $(1,3) \triangleright (3,3)$ □

From the previous proposition, we can deduce a necessary condition for $(3,3)$ outperforming both other regimes $(1,3)$ and $(3,1)$: $\gamma_1 \leq \delta_2$ and $\gamma_2 \leq \delta_1$. We leave to the reader to verify, that the previous also implies $\frac{\gamma_1}{1+\delta_1} \leq \frac{\delta_2}{1+\gamma_2}$ and $\frac{\gamma_2}{1+\delta_2} \leq \frac{\delta_1}{1+\gamma_1}$. Under such conditions, the performances of each regime, can be simplified as:

$$\epsilon_{(1,3)} = \log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right) + \log_2(1+\gamma_2)$$

$$\epsilon_{(3,1)} = \log_2\left(1+\frac{\gamma_2}{1+\delta_2}\right) + \log_2(1+\gamma_1)$$

$$\begin{aligned}\epsilon_{(3,3)} &= \log_2\left(1+\min\left[\gamma_1, \tfrac{\delta_2}{1+\gamma_2}\right]\right) \\ &+ \log_2\left(1+\min\left[\gamma_2, \tfrac{\delta_1}{1+\gamma_1}\right]\right)\end{aligned}$$

According to the expression of $\epsilon_{(3,3)}$, 4 sub-cases have to be considered (2 for each min term). After a quick study, one can define an equivalent criterion for $(3,3) \triangleright (1,3)$ and $(3,3) \triangleright (3,1)$.

Proposition V.2. *When $\gamma_2 \leq \delta_1$ and $\gamma_1 \leq \delta_2$, the two following statements hold:*

$$\left(1+\frac{\delta_2}{1+\gamma_2}\right)(1+\delta_1) \geq (1+\gamma_1)(1+\gamma_2) \Leftrightarrow (3,3) \triangleright (1,3)$$

$$\left(1+\frac{\delta_1}{1+\gamma_1}\right)(1+\delta_2) \geq (1+\gamma_1)(1+\gamma_2) \Leftrightarrow (3,3) \triangleright (3,1)$$

Proof. In this proof, we focus only on the study of $(3,3) \triangleright (1,3)$. In a symmetric way, the second statement can be demonstrated.

Let us first focus on the cases where $\gamma_1 \leq \frac{\delta_2}{1+\gamma_2}$ or $\gamma_2 \leq \frac{\delta_1}{1+\gamma_1}$. For those configurations, the expression of $\epsilon_{(3,3)}$ is rather simple and can be easily compared to those of (1,3) and (3,1), showing immediately that $(3,3) \triangleright (1,3)$. The main difficulty lies in the case where $\frac{\gamma_1}{1+\delta_1} \leq \frac{\delta_2}{1+\gamma_2} \leq \gamma_1 \leq \delta_2$ and $\frac{\gamma_2}{1+\delta_2} \leq \frac{\delta_1}{1+\gamma_1} \leq \gamma_2 \leq \delta_1$. In this last configuration, we get:

$$\begin{aligned}(3,3) \triangleright (1,3) &\Leftrightarrow \epsilon_{(3,3)} - \epsilon_{(1,3)} \geq 0 \\ \Leftrightarrow \frac{(1+\gamma_2+\delta_2)(1+\gamma_1+\delta_1}{(1+\gamma_1)(1+\gamma_2)} &\geq \frac{(1+\gamma_2)(1+\gamma_1+\delta_1)}{(1+\delta_1)} \\ \Leftrightarrow \left(1+\frac{\delta_2}{1+\gamma_2}\right)(1+\delta_1) &\geq (1+\gamma_1)(1+\gamma_2)\end{aligned}$$

A necessary and sufficient condition for $(3,3) \triangleright (1,3)$ is then given by $\gamma_1 \leq \frac{\delta_2}{1+\gamma_2}$ or $\gamma_2 \leq \frac{\delta_1}{1+\gamma_1}$ or $\left(1+\frac{\delta_2}{1+\gamma_2}\right)(1+\delta_1) \geq (1+\gamma_1)(1+\gamma_2)$. However, this sufficient condition can be simplified, since $\gamma_1 \leq \frac{\delta_2}{1+\gamma_2}$ or $\gamma_2 \leq \frac{\delta_1}{1+\gamma_1} \Rightarrow \left(1+\frac{\delta_2}{1+\gamma_2}\right)(1+\delta_1) \geq (1+\gamma_1)(1+\gamma_2)$. Indeed,

$$\begin{cases} \gamma_1 \leq \frac{\delta_2}{1+\gamma_2} \\ \gamma_2 \leq \delta_1 \end{cases} \Rightarrow \begin{cases} \frac{(1+\frac{\delta_2}{1+\gamma_2})}{1+\gamma_1} \geq 1 \\ \frac{1+\gamma_2}{1+\delta_1} \leq 1 \end{cases}$$

$$\Rightarrow \left(1+\frac{\delta_2}{1+\gamma_2}\right)(1+\delta_1) \geq (1+\gamma_1)(1+\gamma_2)$$

In a similar way

$$\gamma_2 \leq \frac{\delta_1}{1+\gamma_1} \Rightarrow \frac{(1+\gamma_2)}{(1+\delta_1)} \leq \frac{(1+\gamma_1+\delta_1)}{(1+\gamma_1)(1+\delta_1)} = \frac{1+\frac{\gamma_1}{1+\delta_1}}{(1+\gamma_1)}$$

And since, $\frac{\gamma_1}{1+\delta_1} \leq \frac{\delta_2}{1+\gamma_2}$, we get $\frac{(1+\gamma_2)}{(1+\delta_1)} \leq \frac{(1+\frac{\delta_2}{1+\gamma_2})}{(1+\gamma_1)}$. Which, means $\left(1+\frac{\delta_2}{1+\gamma_2}\right)(1+\delta_1) \geq (1+\gamma_1)(1+\gamma_2)$.

In the end, we get that, when $\gamma_2 \leq \delta_1$ and $\gamma_1 \leq \delta_2$, $\left(1+\frac{\delta_2}{1+\gamma_2}\right)(1+\delta_1) \geq (1+\gamma_1)(1+\gamma_2) \Leftrightarrow (3,3) \triangleright (1,3)$.

□

VI. Appendix : Best performance SNR/INR region for $(1,1)$

In this section, we focus on defining the SNRs/INRs region $\Omega_{(1,1)}$ in which (1,1) outperforms all the other interferences regimes.

Proposition VI.1.

$$(1,1) \textit{ best regime } \Leftrightarrow \gamma_1 \geq \delta_2(1+\delta_1) \textit{ and } \gamma_2 \geq \delta_1(1+\delta_2)$$

Proof. From Proposition IV.1, we know, that:

$$\begin{cases} (1,1) \rhd (1,3) \\ (1,1) \rhd (3,1) \\ (1,1) \rhd (3,3) \end{cases} \Leftrightarrow \begin{cases} \gamma_1 \geq \delta_2(1+\delta_1) \\ \gamma_2 \geq \delta_1(1+\delta_2) \\ (1,1) \rhd (3,3) \end{cases}$$

We also know that a sufficient condition for $(1,3) \rhd (3,3)$ is $\gamma_2 \geq \delta_1$. Respectively, a sufficient condition for $(3,1) \rhd (3,3)$ is $\gamma_1 \geq \delta_2$ From the previous, we notice that $(1,1) \rhd (1,3)$ does imply $(1,3) \rhd (3,3)$, and by transitivity, we can state that $(1,1) \rhd (3,3)$. From this, we can state that:

$$(1,1) \text{ best regime } \Leftrightarrow \begin{cases} (1,1) \rhd (1,3) \\ (1,1) \rhd (3,1) \end{cases} \Leftrightarrow \begin{cases} \gamma_1 \geq \delta_2(1+\delta_1) \\ \gamma_2 \geq \delta_1(1+\delta_2) \end{cases}$$

□

VII. Appendix: Best performance SNR/INR region for $(3,3)$

In this section, we focus on defining the SNRs/INRs region $\Omega_{(3,3)}$ in which (3,3) outperforms all the other interferences regimes.

Proposition VII.1. *A necessary and sufficient condition for (3,3) outperforming all the other regimes is:*

$$\begin{cases} \gamma_2 \leq \delta_1 \\ \textit{and } \gamma_1 \leq \delta_2 \\ \textit{and } \left(1+\frac{\delta_2}{1+\gamma_2}\right)(1+\delta_1) \geq (1+\gamma_1)(1+\gamma_2) \\ \textit{and } \left(1+\frac{\delta_1}{1+\gamma_1}\right)(1+\delta_2) \geq (1+\gamma_1)(1+\gamma_2) \end{cases} \quad (1)$$

If any of those four conditions are not verified, then $(3,3)$ is outperformed by either $(1,3)$ or $(3,1)$.

Proof. From the previous Propositions V.1 and V.2, we know that,

- a sufficient condition for $(1,3) \rhd (3,3)$ is $\gamma_2 \geq \delta_1$.
- a sufficient condition for $(3,1) \rhd (3,3)$ is $\gamma_1 \geq \delta_2$.
- when $\gamma_2 \leq \delta_1$ and $\gamma_1 \leq \delta_2$, $\left(1+\frac{\delta_2}{1+\gamma_2}\right)(1+\delta_1) \geq (1+\gamma_1)(1+\gamma_2) \Leftrightarrow (3,3) \rhd (1,3)$.
- when $\gamma_2 \leq \delta_1$ and $\gamma_1 \leq \delta_2$, $\left(1+\frac{\delta_1}{1+\gamma_1}\right)(1+\delta_2) \geq (1+\gamma_1)(1+\gamma_2) \Leftrightarrow (3,3) \rhd (3,1)$.

At last, it is also easy to verify, using Proposition IV.1, that $\gamma_2 \leq \delta_1$ and $\gamma_1 \leq \delta_2$ is sufficient for $(3,3) \rhd (1,1)$. □

VIII. Appendix : Best performance regions for $(1,3)$ and $(3,1)$

From the previous Propositions VI.1 and VII.1, we can define the SNRs/INRs regions where $(1,1)$ and $(3,3)$ are the regimes granting the best achievable spectral efficiency. However, it remains a set of values $(\gamma_1,\gamma_2,\delta_1,\delta_2)$, that we denote Ω', for which both $(1,1)$ and $(3,3)$ are outperformed by either $(1,3)$ or $(3,1)$. In those configurations, we have to compare the performance of the two remaining regimes (1,3) and (3,1), in order to determine which one performs the best.

Proposition VIII.1. *If $(\gamma_1,\gamma_2,\delta_1,\delta_2) \in \Omega'$, then*

- $\gamma_1 \leq \delta_2(1+\delta_1)$ *and* $\gamma_2 \geq \delta_1(1+\delta_2)$
- $\gamma_2 \geq \delta_1$ *and* $\gamma_1 \leq \delta_2$

are two sufficient conditions for $(1,3) \rhd (3,1)$.

In a similar way, if $(\gamma_1,\gamma_2,\delta_1,\delta_2) \in \Omega'$, then

- $\gamma_1 \geq \delta_2(1+\delta_1)$ *and* $\gamma_2 \leq \delta_1(1+\delta_2)$
- $\gamma_2 \leq \delta_1$ *and* $\gamma_1 \geq \delta_2$

are two sufficient conditions for $(3,1) \rhd (1,3)$.

Proof. From the previous results, one can obtain sufficient conditions by using the transitivity property of the $\rhd$ operator. The proof is given for $(1,3) \rhd (3,1)$, but can be easily transposed to $(3,1) \rhd (1,3)$.

A sufficient condition for $(1,3) \rhd (3,1)$ is $(1,3) \rhd (1,1)$ and $(1,1) \rhd (3,1)$, which is strictly equivalent to the first statement, according to IV.1. Same way, a sufficient condition for $(1,3) \rhd (3,1)$ is $(1,3) \rhd (3,3)$ and $(3,3) \rhd (3,1)$, which is strictly equivalent to the second statement, according to V.1. □

With Proposition VIII.1, we have covered all Ω' expect two regions, denoted Ω_A and Ω_B:

A. $\Omega_A = \Big\{\gamma_1,\gamma_2,\delta_1,\delta_2 \mid \frac{\gamma_1}{1+\delta_1} \leq \frac{\delta_2}{1+\gamma_2} \leq \gamma_1 \leq \delta_2$ and $\frac{\gamma_2}{1+\delta_2} \leq \frac{\delta_1}{1+\gamma_1} \leq \gamma_2 \leq \delta_1$ and $\left(1+\frac{\delta_2}{1+\gamma_2}\right)(1+\delta_1) \leq (1+\gamma_1)(1+\gamma_2)$ and $\left(1+\frac{\delta_1}{1+\gamma_1}\right)(1+\delta_2) \leq (1+\gamma_1)(1+\gamma_2)\Big\}$.

B. $\Omega_B = \Big\{\gamma_1,\gamma_2,\delta_1,\delta_2 \mid \frac{\delta_2}{1+\gamma_2} \leq \frac{\gamma_1}{1+\delta_1} \leq \gamma_1 \leq \delta_2$ and $\frac{\delta_1}{1+\gamma_1} \leq \frac{\gamma_2}{1+\delta_2} \leq \gamma_2 \leq \delta_1\Big\}$

For these two remaining regions A. and B., we formulate two propositions.

Proposition VIII.2. *When $(\gamma_1,\gamma_2,\delta_1,\delta_2) \in \Omega_B$,*

$$(1,3) \rhd (3,1) \Leftrightarrow \gamma_2 \geq \gamma_1 + (\delta_1 - \delta_2)$$

Proof. When $(\gamma_1,\gamma_2,\delta_1,\delta_2) \in \Omega_B$,

$$\epsilon_{(1,3)} = \log_2(1+\gamma_2) + \log_2\left(1+\frac{\delta_2}{1+\gamma_2}\right) = \log_2(1+\gamma_2+\delta_2)$$
$$\epsilon_{(3,1)} = \log_2(1+\gamma_1) + \log_2\left(1+\frac{\delta_1}{1+\gamma_1}\right) = \log_2(1+\gamma_1+\delta_1)$$

From which, we immediately get:

$$(1,3) \rhd (3,1) \Leftrightarrow \epsilon_{(1,3)} \geq \epsilon_{(3,1)} \Leftrightarrow \gamma_2 + \delta_2 \geq \gamma_1 + \delta_1$$

□

Proposition VIII.3. *When* $(\gamma_1, \gamma_2, \delta_1, \delta_2) \in \Omega_A$,

$$(1,3) \triangleright (3,1) \Leftrightarrow\Leftrightarrow (1+\gamma_1+\delta_1)\gamma_2\delta_2 \geq (1+\gamma_2+\delta_2)\gamma_1\delta_1$$

Proof. When $(\gamma_1, \gamma_2, \delta_1, \delta_2) \in \Omega_A$

$$\begin{array}{l} \epsilon_{(1,3)} = \log_2(1+\gamma_2) + \log_2\left(1+\frac{\gamma_1}{1+\delta_1}\right) \\ \epsilon_{(3,1)} = \log_2(1+\gamma_1) + \log_2\left(1+\frac{\gamma_2}{1+\delta_2}\right) \end{array}$$

By definition,

$$\begin{array}{l} (1,3) \triangleright (3,1) \Leftrightarrow \epsilon_{(1,3)} \geq \epsilon_{(3,1)} \\ \Leftrightarrow \frac{(1+\gamma_2)(1+\gamma_1+\delta_1)}{(1+\delta_1)} \geq \frac{(1+\gamma_1)(1+\gamma_2+\delta_2)}{(1+\delta_2)} \\ \Leftrightarrow (1+\gamma_1+\delta_1)\gamma_2\delta_2 \geq (1+\gamma_2+\delta_2)\gamma_1\delta_1 \end{array}$$

□

To sum up, let us consider this last proposition

Proposition VIII.4. *When* $(\gamma_1, \gamma_2, \delta_1, \delta_2) \in \Omega'$,

$$\begin{array}{l} (1,3) \text{ Best regime } \Leftrightarrow \\ \left\{ \begin{array}{l} [\gamma_2 \geq \delta_1 \text{ and } \gamma_2 \geq \gamma_1 + (\delta_1 - \delta_2)] \\ \text{ or } [\gamma_2 \leq \delta_1 \text{ and } (1+\gamma_1+\delta_1)\gamma_2\delta_2 \geq (1+\gamma_2+\delta_2)\gamma_1\delta_1] \end{array} \right. \end{array}$$

Proof. Summary of Propositions VIII.3 and VIII.2. □